

# Lorenz Gauge in Quantum Cosmology

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## Abstract

In a path-integral approach to quantum cosmology, the Lorenz gauge-averaging term is studied for Euclidean Maxwell theory on a portion of flat four-space bounded by two concentric three-spheres, but with arbitrary values of the gauge parameter. The resulting set of eigenvalue equations for normal and longitudinal modes of the electromagnetic potential cannot be decoupled, and is here studied with a Green-function method. This means that an equivalent equation for longitudinal modes is obtained which has integro-differential nature, after inverting a differential operator in the original coupled system. A complete calculational scheme is therefore obtained for the one-loop semiclassical evaluation of the wave function of the universe in the presence of gauge fields. This might also lead to a better understanding of how gauge independence is actually achieved on manifolds with boundary, whose consideration cannot be avoided in a quantum theory of the universe.

Over the last decade, much work in one-loop quantum cosmology has been devoted to the analysis of the wave function of the universe in the case of gauge fields on the Euclidean four-ball, or on a portion of flat Euclidean four-space bounded by two concentric three-spheres of radii  $\tau_-$  and  $\tau_+$ . We are here interested in the electromagnetic case, since a proper understanding of this model may already cast new light on quantum cosmology and quantum field theory. The normal and tangential components of the electromagnetic potential are then expanded on a family of concentric three-spheres in the form

$$A_0(x, \tau) = \sum_{n=1}^{\infty} R_n(\tau) Q^{(n)}(x) , \quad (1)$$

$$A_k(x, \tau) = \sum_{n=2}^{\infty} [f_n(\tau) S_k^{(n)}(x) + g_n(\tau) P_k^{(n)}(x)] \quad \text{for all } k = 1, 2, 3 , \quad (2)$$

where  $Q^{(n)}(x)$ ,  $S_k^{(n)}(x)$ ,  $P_k^{(n)}(x)$  are scalar, transverse and longitudinal vector harmonics on  $S^3$ , respectively. Gaussian averages over gauge functionals are then performed according to the Faddeev-Popov scheme, so that the part of the full Euclidean action involving the potential  $A_\mu$  reads [1]

$$\int_M \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{[\Phi(A)]^2}{2\alpha} \right] \sqrt{\det g} d^4x ,$$

where  $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$  denotes the electromagnetic-field tensor,  $g$  is the background four-metric,  $\Phi$  is an arbitrary gauge-averaging functional defined on the space of connection one-forms, and  $\alpha$  is a dimensionless parameter. We are here interested in the Lorenz choice for  $\Phi$ , i.e.

$$\Phi_L \equiv {}^{(4)}\nabla^\mu A_\mu = {}^{(4)}\nabla^0 A_0 + A_0 \text{Tr } K + {}^{(3)}\nabla^i A_i , \quad (3)$$

where  $K$  is the extrinsic-curvature tensor of the three-sphere boundary  $S^3$ , and  ${}^{(3)}\nabla$  is the induced connection on  $S^3$ . The transverse modes  $f_n$  are decoupled, whereas longitudinal and normal modes turn out to obey the coupled eigenvalue equations [1]

$$\hat{A}_n g_n(\tau) + \hat{B}_n R_n(\tau) = 0 , \quad (4)$$

$$\hat{C}_n g_n(\tau) + \hat{D}_n R_n(\tau) = 0 , \quad (5)$$

where the differential operators resulting from the choice (3) read

$$\hat{A}_n \equiv \frac{d^2}{d\tau^2} + \frac{1}{\tau} \frac{d}{d\tau} - \frac{1}{\alpha} \frac{(n^2 - 1)}{\tau^2} + \lambda_n , \quad (6)$$

$$\hat{B}_n \equiv (n^2 - 1) \left[ \left( \frac{1}{\alpha} - 1 \right) \frac{d}{d\tau} + \left( \frac{3}{\alpha} - 1 \right) \frac{1}{\tau} \right] , \quad (7)$$

$$\hat{C}_n \equiv \left(1 - \frac{1}{\alpha}\right) \frac{1}{\tau^2} \frac{d}{d\tau} + \frac{2}{\alpha} \frac{1}{\tau^3}, \quad (8)$$

$$\hat{D}_n \equiv \frac{1}{\alpha} \frac{d^2}{d\tau^2} + \frac{3}{\alpha} \frac{1}{\tau} \frac{d}{d\tau} - \left(\frac{3}{\alpha} + n^2 - 1\right) \frac{1}{\tau^2} + \lambda_n, \quad (9)$$

$\lambda_n$  being the eigenvalues [1]. For arbitrary values of  $\alpha$  one cannot decouple Eqs. (4) and (5) and map them into another system involving only differential operators. One can however use with some profit integral equations and Green-kernel methods. For this purpose, we have to impose a suitable set of boundary conditions. Since the tangential components of  $A_\mu$  and the gauge-averaging functional should vanish at the boundary to achieve gauge invariance of the whole set of boundary conditions [1,2], we have

$$g_n(\tau_+) = g_n(\tau_-) = 0, \quad (10)$$

$$\left[ \frac{dR_n}{d\tau} + \frac{3}{\tau} R_n \right]_{\tau=\tau_+} = \left[ \frac{dR_n}{d\tau} + \frac{3}{\tau} R_n \right]_{\tau=\tau_-} = 0. \quad (11)$$

These boundary conditions tell us how to proceed in order to solve the system of coupled equations (4) and (5). Since we are aiming to invert differential operators, it is clear that we have to consider operators having well defined inverses once such boundary conditions are assigned. Bearing this in mind, and denoting by  $\hat{A}_n^{-1}$  the inverse of  $\hat{A}_n$ , Eq. (4) leads to

$$g_n = -\hat{A}_n^{-1} \hat{B}_n R_n, \quad (12)$$

so that Eq. (5) implies

$$R_n = \mathcal{P}_n R_n, \quad (13)$$

having defined

$$\mathcal{P}_n \equiv \hat{D}_n^{-1} \hat{C}_n \hat{A}_n^{-1} \hat{B}_n. \quad (14)$$

On denoting by  $P_n(\tau, y)$  the kernel of  $\mathcal{P}_n$ , and by  $G_n(\tau, y)$  the Green kernel of  $\hat{A}_n$ , we therefore find (see below for the notation)

$$R_n(\tau) = \int_{\tau_-}^{\tau_+} P_n(\tau, y) R_n(y) dy, \quad (15)$$

$$g_n(\tau) = - \int_{\tau_-}^{\tau_+} G_n(\tau, y) (\hat{B}_n R_n)(y) dy. \quad (16)$$

The general form of the kernels is obtained with the help of standard techniques. For example,  $G_n$  satisfies the differential equation

$$\hat{A}_n G_n(\tau, y) = 0 \quad \forall \tau \neq y, \quad (17)$$

as well as the continuity condition

$$\lim_{\tau \rightarrow y^+} G_n(\tau, y) - \lim_{\tau \rightarrow y^-} G_n(\tau, y) = 0, \quad (18)$$

and the jump condition

$$\lim_{\tau \rightarrow y^+} \frac{\partial}{\partial \tau} G_n(\tau, y) - \lim_{\tau \rightarrow y^-} \frac{\partial}{\partial \tau} G_n(\tau, y) = 1. \quad (19)$$

Moreover, by virtue of (10),  $G_n(\tau, y)$  obeys the boundary conditions

$$G_n(\tau_-, y) = 0, \quad (20)$$

$$G_n(\tau_+, y) = 0. \quad (21)$$

On defining  $\tau_- \equiv \min(\tau, y)$  and  $\tau_+ \equiv \max(\tau, y)$ , the general theory of one-dimensional boundary-value problems [3] makes it therefore possible to express the Green kernel  $G_n(\tau, y)$  in the form

$$G_n(\tau, y) = C \frac{u_1(M\tau_-)u_2(M\tau_+)}{W(u_1, u_2; \xi)}, \quad (22)$$

where  $C$  is a constant,  $\nu \equiv \sqrt{\frac{n^2-1}{\alpha}}$ , the parameter  $M$  is the square root of  $\lambda_n$ ,  $u_1$  and  $u_2$  are linearly independent solutions of  $\hat{A}_n u = 0$  vanishing at  $\tau_-$  and  $\tau_+$ , respectively, and having Wronskian  $W(u_1, u_2; \xi)$ . They can be chosen in the form

$$u_1(\tau) = J_\nu(M\tau), \quad (23)$$

$$u_2(\tau) = \tilde{A}J_\nu(M\tau) + \tilde{B}N_\nu(M\tau). \quad (24)$$

The formula (22) should be inserted into Eq. (16) where  $R_n$  (obtained from (15)) reads, more explicitly,

$$R_n(\tau) = \int_{\tau_-}^{\tau_+} dy \Gamma_n(\tau, y) \int_{\tau_-}^{\tau_+} dz (\hat{C}_n G_n(y, z)) (\hat{B}_n R_n)(z), \quad (25)$$

having denoted by  $\Gamma_n(\tau, y)$  the Green kernel of  $\hat{D}_n$ .

We have therefore outlined a complete computational scheme for the evaluation of gauge modes in quantum cosmology when arbitrary values of the gauge parameter  $\alpha$  are considered. Interestingly, integral equations and Green-kernel methods seems to be unavoidable technical steps if the Faddeev–Popov path integral for the wave functional is studied for all  $\alpha$ . In other words, even if local boundary conditions are imposed with linear covariant gauges, the contribution to functional determinants resulting from longitudinal and normal modes involves, in general, a non-local analysis, as is clear from the integral formulae (25) and (16). It now remains to be seen whether the full  $\zeta(0)$  value [1] is independent of  $\alpha$ . Although the question remains unsettled, Eqs. (25) and (16) seem to provide new tools for the solution of this longstanding problem in one-loop quantum cosmology. It should also be stressed that gauge-invariant boundary conditions make it necessary to invert operators of Bessel type (see (6) and (9)). In this sense, Bessel functions remain the fundamental tool for a flat-space analysis, even when gauge modes remain coupled.

## REFERENCES

- [1] G. Esposito, A. Yu. Kamenshchik and G. Pollifrone, *Euclidean Quantum Gravity on Manifolds with Boundary*, Fundamental Theories of Physics, Vol. 85 (Kluwer, Dordrecht, 1997).
- [2] I. G. Avramidi and G. Esposito, Commun. Math. Phys. **200**, 495 (1999).
- [3] I. Stakgold, *Green's Functions and Boundary Value Problems* (Wiley, New York, 1979).